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UNIFORMLY ELLIPTIC OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract

Given a Riemannian manifold (M, g), we study the solutions of heat equations associated with second order differential operators in divergence form that are uniformly elliptic with respect to g. Typical examples of such operators are the Laplace operators of Riemannian structures which are quasi-isometric to g. We first prove some Poincaré and Sobolev inequalities on geodesic balls. Then we use Moser's iteration to obtain Harnack inequalities. Gaussian estimates, uniqueness theorems, and other applications are also discussed. These results involve local or global lower bound hypotheses on the Ricci curvature of g. Some of them are new even when applied to the Laplace operator of (M, g).

1. Introduction

Let (M, g) be a complete Riemannian manifold. In this paper, we study some second-order differential operators which are to g what uniformly elliptic operators in divergence form are to the Euclidean metric on $\mathbb{R}^n$. The simplest example of such an operator is the Laplace-Beltrami operator Δ associated with g. Under lower bound hypotheses on the Ricci curvature, Li and Yau have obtained definitive results concerning Δ and the corresponding heat equation. By rather elementary means, they proved in [28] a remarkable gradient estimate which implies parabolic Harnack inequalities as well as upper and lower Gaussian bounds for the kernel of the heat flow semigroup $e^{-t\Delta}$. In fact, the parabolic gradient estimate of Li-Yau is a generalization of previous elliptic results by Yau [43] and Cheng-Yau [10]. One important aspect of these works is that they give global estimates when the Ricci curvature is nonnegative on M. This is well illustrated by a well-known result of Yau who proved in [43] that any manifold (M, g) with nonnegative Ricci curvature has the strong Liouville property (i.e., any positive harmonic function on (M, g) is constant).

Let us now introduce a typical example of the operators studied in this paper. Let $\widetilde{\Delta}$ be the Laplacian associated with another metric $\tilde{g}$ on M

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such that, for some $\alpha \geq 1$,

$$\alpha^{-1}\tilde{g} \leq g \leq \alpha \tilde{g}.$$

The point is that we want to study $\widetilde{\Delta}$ under lower bound hypotheses on the Ricci curvature of the fixed metric g. This means that we have no control on the Ricci curvature of $\tilde{g}$, and consequently, that the methods of [28] do not apply. However, we are going to prove parabolic Harnack inequalities for the solutions of the heat equation $(\partial_t + \widetilde{\Delta})u = 0$ and Gaussian estimates for the kernel of the semigroup $e^{-t\widetilde{\Delta}}$. These results are global when the Ricci curvature of g is nonnegative on M. For instance, we prove that $(M, \tilde{g})$ has the strong Liouville property for any $\tilde{g}$ as above whenever g has nonnegative Ricci curvature. This extends Yau's result. It is known that, in general, the (strong) Liouville property is not stable under such quasi-isometric changes of metrics (see [30]).

The point of view adopted in this paper is to consider $\widetilde{\Delta}$ as a uniformly elliptic, divergence form operator with respect to g. In $\mathbb{R}^n$, a divergence form operator L (of the simplest kind) is a second order differential operator of the type

$$L = -\sum_{i,j=1}^{n} \partial_i a_{i,j} \partial_j, \qquad a_{i,j} = a_{j,i},$$

and it is uniformly elliptic if the matrix-valued function $(a_{i,j}(t, x))$ satisfies for some $\alpha \geq 1$ and for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$\alpha^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(t, x)\xi_i \xi_j \leq \alpha |\xi|^2.$$

The study of the parabolic equation $(\partial_t + L)u = 0$ has a long history marked by works of Nash [34], Moser [31]–[33], Aronson [4], and others. The Liouville property is proved in [31], the parabolic Harnack inequality in [32], [33], and Gaussian bounds in [4]. The pioneering work of Nash [34] contains ideas that also lead to these results (see [18]). Our approach in this paper is to use Moser's method, which clearly extends quite generally as soon as its basic ingredients are available. These ingredients are some precise Sobolev and Poincaré inequalities on balls.

The main contribution of this paper is to remark that the gradient estimates of Yau and his collaborators imply some Sobolev and Poincaré inequalities on balls which are good enough to run Moser's method in an optimal way. On the one hand, this should come as no surprise. Li-Yau's paper [28], as well as previous works by Yau and others, have made clear that a lower bound hypothesis on the Ricci curvature provides enough information to develop an analysis on (M, g). On the other hand, previous attempts to use Moser's method in this setting, as in [9] and [8], have not yielded optimal results.

The results in [9] and [8] depend only on the part of Moser's method that deals with upper bounds and uses the Sobolev inequality. They are not sharp because they rest on a nonsharp Sobolev inequality which follows from the work of Croke [11]. What seems to have been missing at that time, and is now available thanks to the work of Varopoulos [40], is the understanding of the links between Sobolev inequalities and semigroup estimates. Indeed, using Varopoulos results, we show that optimal Sobolev inequalities on balls can easily be obtained from the gradient estimate of Cheng-Yau (which goes back to 1975).

Roughly speaking, the Poincaré inequality enters Moser's method in order to obtain lower bounds on positive solutions. When dealing with elliptic equations, a classical Poincaré inequality on balls is needed. In our setting, such an inequality has been obtained by Buser in [7]. However, if we insist on treating parabolic equations as well, a more involved variation on Poincaré inequality is needed (see [32]). As in [36], we show that the relevant inequality follows from the classical one by adapting an idea of Jerison [21].

The content of this paper can be seen from different points of view. The Sobolev and Poincaré inequalities presented in §3 and proved in §§9 and 10 are of independent interest. In §4 we introduce a natural class of uniformly elliptic divergence form operators, and in §5 we state the powerful Harnack-Moser inequalities which follow from the Sobolev and Poincaré inequalities and Moser iteration. §§6, 7, and 8 present applications of Harnack-Moser inequalities. Some of the results obtained in §§7 and 8 improve upon previous ones even when applied to the Laplace operator Δ on (M, g). Notation is introduced in §2, where we recall some well-known volume estimates that play a basic part in this work as well as the gradient inequality of Cheng-Yau which is used in the proof of Sobolev inequalities.

Some of the results presented in this paper have been announced in [35], where a special emphasis is placed on the case when (M, g) has nonnegative Ricci curvature.

2. Preliminaries

Here we present the notation used in this paper. Denote by ∇f the gradient induced by the fixed metric g on M, and set $|\nabla f|^2 = g(\nabla f, \nabla f)$.

Let Δ be the Laplace-Beltrami operator on (M, g) with the convention that $(\Delta f, f) = \int |\nabla f|^2 dv$, where v is the Riemannian volume, and $(f_1, f_2) = \int f_1 f_2 dv$. In other words, Δ is a positive operator on $L^2(M, dv)$. The operator Δ can be written as $\Delta f = -\operatorname{div}(\nabla f)$ where $\operatorname{div}(X)$ is defined, for any vector field X, by $\int Xf\, dv = -\int f \operatorname{div}(X)\, dv$.

Denote by T_x the tangent space at x and by T_M the tangent bundle. We denote by $\operatorname{End}(T_M)$ and $\operatorname{Sym}(T_M)$ the bundle of endomorphisms and the bundle of symmetric endomorphisms, respectively, over M. Also let Ric be the Ricci curvature tensor of g.

Let ρ be the Riemannian distance on M. Given a ball $B = B(x, r)$, we denote by $V(x, r) = V = v(B)$ its volume, and by $sB = B(x, sr)$ the concentric ball with radius sr. We define $K = K(B) = K(x, r)$ to be the smallest nonnegative number such that $\mathrm{Ric} \geq -Kg$ in $2B = B(x, 2r)$. Note that K is a lower bound of the Ricci curvature in $2B$ instead of B. However, the number 2 has been chosen for convenience and can be replaced by any fixed number strictly greater than 1.

Among the basic tools used in this paper are classical volume estimates which we now recall. Let $V_a(r)$ be the volume of the ball of radius r in the simply connected space with constant sectional curvature equal to $-a/(n-1)$. In the sequel $a = K$ is a nonnegative number, and the above space is a hyperbolic space if $a > 0$ and the Euclidean space if $a = 0$. In any case, we have

$$\omega_n s^n \leq V_a(s) \leq \omega_n s^n \exp(\sqrt{(n-1)a}s), \qquad s > 0, a \geq 0,$$

where ω_n is the volume of the Euclidean ball of radius one. For any ball $B = B(x, r) \subset M$ it is well known that we have (see [8])

$$V(x, s)/V(x, s') \leq V_K(s)/V_K(s'), \qquad 0 < s' < s < 2r.$$

From the above, we deduce

$$(1) \qquad V(x, s) \leq \omega_n s^n \exp(\sqrt{(n-1)K}s), \qquad 0 < s < 2r,$$

and

$$(2) \quad V(x, s) \leq V(x, s')(s/s')^n \exp(\sqrt{(n-1)K}s), \qquad 0 < s' < s < 2r,$$

which implies, for $0 < s' < s < r$, $y \in B$, and setting $\rho(x, y) = \rho$,

$$(3) \qquad V(y, s) \leq V(x, s')((\varphi + s)/s')^n \exp(\sqrt{(n-1)K}(\rho + s)).$$

These estimates are used again and again in this paper.

We now recall the basic gradient estimate on which this work rests. In [43], Yau proved that any positive solution u of $\Delta u = 0$ on a manifold M with Ricci curvature bounded from below by $-K$ satisfies

$$|\nabla \ln(u)| \leq \sqrt{(n-1)K}.$$

In [10], Cheng and Yau proved a local version which reads as follows. Fix $0 < \delta < 1$. There exists a constant C depending only on n and δ such that any positive solution u of $\Delta u = 0$ in $2B = B(x, 2r) \subset M$ satisfies

$$|\nabla \ln(u)| \leq C(r^{-1} + \sqrt{K}) \quad \text{in } 2\delta B. \tag{4}$$

Integrating along minimal paths, an elliptic Harnack inequality can be obtained. Namely, any u as above satisfies

$$u(z)/u(y) \leq e^{C(1+\sqrt{K}r)}, \qquad z, y \in 2\delta B.$$

In [28], Li and Yau generalized the above to the parabolic equation $(\partial_t + \delta)u = 0$. However, we will only need the elliptic version.

Let $H_t = e^{-t\Delta}$ be the heat flow semigroup on M, and let h_t be its kernel. We consider also the Poisson semigroup $P_t = e^{-t\sqrt{\Delta}}$ which can be obtained from H_t through the subordination formula

$$P_t = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-s} H_{t^2/4s}\, ds, \qquad t > 0. \tag{5}$$

We denote the kernel of P_t by p_t. The following estimates are easy consequences of the Cheng-Yau inequality.

Lemma 2.1. *For any ball $B = B(x, r)$, the kernels p_t and h_t satisfy*

$$p_r(x, x), h_{r^2}(x, x) \leq e^{C(1+\sqrt{K}r)} V^{-1},$$

where C depends only on n.

Note that the upper bound on h_{r^2} which follows from [28] reads

$$h_{r^2}(x, x) \leq e^{C(1+Kr^2)} V^{-1}.$$

In order to obtain the improved estimate stated in the lemma, we remark that $h_t(x, x)$ is a decreasing function of t. Hence, using (5) we have

$$p_r(x, x) \geq \frac{1}{\sqrt{\pi}} \int_1^\infty s^{-1/2} e^{-s} h_{r^2/s}(x, x)\, ds \geq c h_{r^2}(x, x),$$

and the upper estimate of h_{r^2} follows from the upper estimate of p_r. Note that corresponding lower bounds hold as well (however, the lower bound for h_{r^2} is of the form $e^{-C(1+Kr^2)} V^{-1}$). Lemma 2.1 will be used in §10 where various Sobolev inequalities are proved.

3. Sobolev and Poincaré inequalities on balls

The role played by Sobolev and Poincaré inequalities in analysis and geometry is well known and a fair amount of work has been devoted to their study. All the results in this paper rest on versions of these inequalities given by Theorems 3.1 and 3.2 below. The proofs of these inequalities are given in §§9 and 10. We start by reviewing some known inequalities which are closely related to the ones obtained in this paper.

Given a compact manifold M it is not very difficult to see that there exist $P(M)$ and $S(M)$ such that $\|f - f_M\|_2^2 \leq P(M)\|\nabla f\|_2^2$ and $\|f - f_M\|_{2q}^2\, dv \leq S(M)\|\nabla f\|_2^2\, dv$ for $f \in \mathscr{C}^\infty(M)$, where $q = n/(n-2)$ (for simplicity we suppose here that $n > 2$). Note that $P(M)$ can always be estimated by $P(M) \leq V(M)^{(1-1/q)}S(M) = V(M)^{-2/n}S(M)$ (here, $V(M)$ denotes the volume of M). What is more difficult, but by now well known, is that $P(M)$ and $S(M)$ can be estimated in terms of the diameter $d = d(M)$, the volume $V = V(M)$, and a lower bound $-K$ $(K \geq 0)$ of the Ricci curvature. Namely, it was proved by Li and Yau in [27] that $P(M) \leq e^{C_n(1+\sqrt{K}d)}d^2$, and by S. Gallot in [19] that $S(M) \leq e^{C_n(1+\sqrt{K}d)}d^2V^{-2/n}$. Apart from the dimensional constants, these estimates are sharp.

Now, let $B = B(x, r)$ be a ball in a complete manifold M. In view of the above, it is natural to conjecture that

$$(6) \qquad \int_B |f - f_B|^2\, dv \leq e^{C_n(1+\sqrt{K}r)}r^2 \int_B |\nabla f|^2\, dv, \qquad f \in \mathscr{C}^\infty(B),$$

and

$$(7) \qquad \int_B |f - f_B|^{2q}\, dv \leq e^{C_n(1+\sqrt{K}r)}r^2V^{-2/n} \int_B |\nabla f|^2\, dv, \qquad f \in \mathscr{C}^\infty(B),$$

where, according to our notation, $K = K(B) \geq 0$ is such that $\text{Ric} \geq -Kg$ in $2B$, and V is the volume of B. Indeed, (6) has been proved by P. Buser in [7]. In this paper, we offer a generalization (Theorem 3.2) of (6) which plays an important role in the study of parabolic equations. We also obtain (Theorem 3.1) a slightly weakened version of (7) which is powerful enough for our applications. Although it does not play a role in this paper, it would be interesting to know whether or not (7) holds in general (we are able to prove (7) only under the strong (and unpleasant) hypothesis that B has convex boundary; see §10). We now state the Sobolev inequality on which most of this paper rests.

Theorem 3.1. *For $n > 2$, there exists C, depending only on n, such that for all $B \subset M$ we have*

$$\left(\int |f|^{2q}\,dv\right)^{1/q} \le e^{C(1+\sqrt{K}r)}V^{-2/n}r^2\left(\int (|\nabla f|^2 + r^{-2}|f|^2)\,dv\right),$$
$$f \in \mathscr{C}_0^\infty(B),$$

where $q = n/(n-2)$. For $n \le 2$, the above inequality holds with n replaced by any fixed $n' > 2$.

This result is weaker than (7) in two aspects. First, it does not include any kind of Poincaré inequality. Second, it involves only functions with compact support in B. The proof and some variations on this result (including L^p versions) are presented in §10.

We now pass to a generalization of (6). In fact, Theorem 3.1 and (6) would be sufficient to follow the approach in [31] and study certain elliptic differential equations on M. However, in order to study parabolic equations we need an improved version of (6) (see [32, Lemma 3, p. 120] for a statement in $\mathbb{R}^n$). Let φ be a nonincreasing function from $[0, +\infty[$ to $[0, 1]$ such that $\varphi(t) = 0$ for $t > 1$. We assume that there exists a $\beta > 0$ such that $\varphi(t + (1-t)/2) \ge \beta\varphi(t)$, $1/2 \le t \le 1$, and set, for any ball $B = B(x, r)$, $\Phi_B = \varphi(\rho(x, \cdot)/r)$.

Theorem 3.2. *There exist C as above and C_φ depending only on δ, n, and φ such that, for any ball $B \subset M$, we have*

$$\int |f - f_\Phi|^2\Phi\,dv \le C_\varphi e^{C\sqrt{K}r}r^2\int |\nabla f|^2\Phi\,dv, \qquad f \in \mathscr{C}^\infty(B),$$

where $\Phi = \Phi_B$ and $f_\Phi = \int f\Phi\,dv / \int \Phi\,dv$.

There are also L^p versions of this result (see §9).

4. Uniformly elliptic operators in divergence form

In this section, we introduce the notation concerning the uniformly elliptic operators in divergence form which are studied in this paper. Let us first consider the simplest example of these operators (besides Δ). Namely, let $\tilde{\Delta}$ be the Laplace operator associated with a metric $\tilde{g}$ such that $\alpha^{-1}\tilde{g} \le g \le \alpha\tilde{g}$. Consider the section $\mathscr{A}$ of $\mathrm{Sym}(T_M)$ defined by $\tilde{g}(X, Y) = g(\mathscr{A}^{-1}X, Y)$. With some evident notation, we have $\tilde{\nabla} = \mathscr{A}\nabla$, $g(\mathscr{A}\nabla f, \nabla f) = \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f)$, and

$$\tilde{\Delta} = -\widetilde{\mathrm{div}}(\tilde{\nabla} f) = -m^{-1}\,\mathrm{div}(m\mathscr{A}\nabla f),$$

where $m = \sqrt{\det\mathscr{A}}$. Moreover, the quasi-isometry between g and $\tilde{g}$

implies that $\mathscr{A}$ and m satisfy $\alpha^{-1}|X|^2 \leq g(\mathscr{A}X, X) \leq \alpha|X|^2$ and $\alpha^{-n/2} \leq m \leq \alpha^{n/2}$.

We generalize these considerations as follows. Let $\mathscr{A}_t$ be a measurable section of $\operatorname{End}(T_M)$ depending on the parameter t. In other words, for $t \in \mathbb{R}$ and $x \in M$, $\mathscr{A}_{t,x}$ is an endomorphism of T_x, and the function $(t, x) \to (x, \mathscr{A}_{t,x}) \in \operatorname{End}(T_M)$ is measurable. Let m be a positive measurable function on M. We consider the operator $L = L_{\mathscr{A},m}$ defined by

$$Lu = -m^{-1} \operatorname{div}(m\mathscr{A}\nabla u). \tag{8}$$

We make the quantitative assumption that there exist $\alpha, \mu \geq 1$ such that

$$\mu^{-1} \leq m(x) \leq \mu, \qquad x \in M, \tag{9}$$

and

$$\alpha^{-1}|X|^2 \leq g(\mathscr{A}_{t,x}X, X), \quad |\mathscr{A}_{t,x}X| \leq \alpha|X| \quad \forall t \in \mathbb{R}, x \in M, X \in T_x. \tag{10}$$

Note that, in general, we do not assume that $\mathscr{A}$ is symmetric. However, in the case when $\mathscr{A}$ is symmetric and independent of t, we will consider the metric $\tilde{g}$ defined by $\tilde{g}(X, Y) = g(\mathscr{A}^{-1}X, Y)$ and the associated distance function $\tilde{\rho}$.

Finally, we also consider operators of the form $\mathscr{L} = L+$ lower order terms. Let $\mathscr{X}_t$ and $\mathscr{Y}_t$ be two vector fields on M such that the functions $(t, x) \to (x, \mathscr{X}_{t,x})$ and $(t, x) \to (x, \mathscr{Y}_{t,x})$ are measurable, and let $b(t, x)$ be a measurable function on $\mathbb{R} \times M$. We define $\mathscr{L}$ by

$$\mathscr{L}u = -m^{-1} \operatorname{div}(m(\mathscr{A}\nabla u + u\mathscr{X})) + \mathscr{Y}u + bu.$$

On $\mathscr{X}$, $\mathscr{Y}$ and b we make the assumption that there exists $\beta \geq 0$ such that

$$|\mathscr{X}|^2 + |\mathscr{Y}|^2 + |b| \leq \beta. \tag{11}$$

Under the uniform ellipticity hypotheses (9), (10), (11), we study the equation

$$(\partial_t + \mathscr{L})u = 0. \tag{12}$$

We will refer to $\mathscr{A}$, $\mathscr{X}$, $\mathscr{Y}$, b and m as the coefficients of the above equation (or of the operator $\mathscr{L}$). It is one of the important features of the results of this paper to hold for operators with measurable coefficients satisfying the above uniform ellipticity hypotheses. When studying the solutions of (12) in this generality, it is necessary to specify what "solution" means. The natural notion of weak solution is spelled out in [33] and [5], and it is well known that the study of (12) in the above genrality can be reduced to the case when additional qualitative smoothness assumptions are made on the coefficients of $\mathscr{L}$. Hence, we do assume in all the proofs given in this paper that m, $\mathscr{A}$, $\mathscr{X}$, $\mathscr{Y}$, and b are smooth, and that solution means classical solution. We say that u is a *subsolution* of (12) when $(\partial_t + \mathscr{L})u \leq 0$, and that u is a *supersolution* if $-u$ is a subsolution.

5. Harnack-Moser inequalities

We now follow the method introduced in [32] and [33] by J. Moser, and state some Harnack inequalities for the solutions of the parabolic equation (12). The proofs will be omitted. Note that, by following [33], one avoids using the John-Nirenberg type lemma which is one of the difficult points in [32]. Moser's iteration has two steps. In the first step, mean value type inequalities are obtained by an iterative argument. The main ingredient is the Sobolev inequality of Theorem 3.1. Theorem 5.1 below is the basic result of this first step: it is a powerful tool which yields the Gaussian upper bounds of §6 and all the results discussed in §8. It will be convenient to fix a parameter $0 < \delta < 1$, and to associate with any ball B and any real s the sets

$$Q =]s-r^2, s[\times B, \quad Q_\delta =]s-\delta r^2, s[\times \delta B, \quad Q'_\delta =]s-r^2, s-(1-\delta)r^2[\times \delta B,$$

and, for $0 < \varepsilon < \eta < \delta < 1$,

$$Q_- = [s - \delta r^2, s - \eta r^2] \times \delta B, \qquad Q_+ = [s - \varepsilon r^2, s[\times \delta B.$$

We also set $\|u\|_{p,Q} = (\iint_Q |u(t, x)|^p \, dt \, dv)^{1/p}$ for a function u defined in Q.

Theorem 5.1. *Given $0 < p < +\infty$, any nonnegative subsolution u of (12) in Q satisfies*

$$\sup_{Q_\delta}\{u^p\} \leq C'(1-\delta)^{-(2+n)}(1+\beta r^2)^\gamma e^{C\sqrt{K}r}(r^2 V)^{-1}\|u\|_{p,Q}^p$$

for $0 < \delta < 1$, where $C, \gamma > 0$ depend only on n and p, whereas C' depends also on α and μ. Moreover, when $0 < p \leq 2$, we can take $\gamma = 1 + n/2$, and C depends only on n (when $n \leq 2$, n has to be replaced by any fixed $n' > 2$).

The second step in Moser's method is more technical. Different mean value inequalities are put together to finally obtain the Harnack inequality of Theorem 5.3 below. Here, the basic ingredient is the Poincaré inequality of Theorem 3.2, together with Theorem 5.1 and a variant of it which deals with supersolutions. An intermediate result worth noting reads as follows.

Theorem 5.2. *Let $0 < p_0 < 1 + n/2$ be fixed. Then any nonnegative supersolution u of* (12) *in Q satisfies*

$$\|u\|_{p_0, Q_-} \leq (r^2 V)^{1/p_0} e^{C_0 F(r)} \inf_{Q_+}\{u\},$$

where $F(r) = (1 + \beta r^2)^{3(1+n/2)} e^{C\sqrt{K}r}$, and C depends only on n, whereas C_0 depends on n, α, μ, ε, η, δ, and p_0.

From the above two results and some classical chaining arguments, we obtain

Theorem 5.3. *Any nonnegative solution u of* (12) *in Q satisfies*

$$\sup_{Q_-}\{u\} \leq e^{C(1+\beta r^2 + K r^2)} \inf_{Q_+}\{u\},$$

with C depending only on n, δ, ε, η, α, and μ.

Corollary 5.4. *If u is a positive solution of* (12) *in $]0, T[\times B$, then it satisfies*

$$\ln\left(\frac{u(t', y')}{u(t, y)}\right) \leq C\left(\frac{\rho^2}{(t - t')} + \left(\beta + K + \frac{1}{r^2} + \frac{1}{t'}\right)(t - t')\right)$$

for $0 < t' < t < T$ and $y, y' \in \delta B$, where $\rho = \rho(y, y')$ and C depends only on n, δ, α and μ.

Corollary 5.5. *There exist $0 < \gamma < 1$ and C, depending only on n, α, μ, and δ, such that, for any solution u of* (12) *in Q, we have*

$$|u(t', y') - u(t, y)| \leq C(1 + \sqrt{\beta + K}r)^\gamma (\overline{\rho}/r)^\gamma \|u\|_{\infty, Q},$$

where $\overline{\rho} = \max\{\sqrt{|t - t'|}, \rho(y, y')\}$ and $(t, y), (t', y') \in Q_\delta$.

One of the advantages of Moser's approach is that it applies to a wide range of equations and the above results are by no means the most general that can be obtained. For instance, nonhomogeneous terms can be added, and the coefficients of the lower order terms can be taken to belong to some $L^{p,q}$ classes instead of being bounded. Examples of statements of this kind are given below. As a motivation, we mention that we will encounter nonhomogeneous equations when studying space derivatives of the heat kernel (see Theorem 6.5 below). We follow closely Aronson-Serrin's paper [5] to which we refer the reader for proofs that can be adapted to our setting.

For $(u, t, (x, X)) \in \mathbb{R} \times \mathbb{R} \times T_M$, let $\mathbf{A}(u, t, x, X) \in T_x$ and $\mathbf{B}(u, t, x, X) \in \mathbb{R}$ be such that

$$(u, t, x, X) \to (x, \mathbf{A}) \in T_M,$$
$$(u, t, x, X) \to \mathbf{B}$$

are measurable functions. We consider the equation

$$\begin{aligned} \partial_t u(t, x) = & -m \operatorname{div}\{m\mathbf{A}(u(t, x), t, x, \nabla u(t, x))\} \\ & + \mathbf{B}(u(t, x), t, x, \nabla u(t, x)), \end{aligned} \tag{13}$$

where m is as before. On $\mathbf{A}$ and $\mathbf{B}$ we make the hypotheses that

$$g(X, \mathbf{A}(u, t, x, X)) \geq \alpha^{-1}|X|^2 - a_1^2 u^2 - a_2^2,$$
$$|\mathbf{A}(u, t, x, X)| \leq \alpha|X| + a_3|u| + a_4,$$
$$|\mathbf{B}(u, t, x, X)| \leq a_5|X| + b_1 + b_2,$$

where $\alpha \geq 1$ is a constant, and the a_i's and b_i's are nonnegative functions of (t, x) that belong to some classes $L^{p,q}_{\text{loc}}(\mathbb{R} \times M)$ with

$$p \geq 2/(1-\theta) \quad \text{and} \quad n/2p + 1/q \leq (1-\theta)/2 \quad \text{for the } a_i\text{'s},$$
$$p \geq 1/(1-\theta) \quad \text{and} \quad n/2p + 1/q \leq (1-\theta) \quad \text{for the } b_i\text{'s}$$

for some fixed $0 < \theta \leq 1$ (here, as before, p corresponds to the space variable x and q to the time variable t). When studying the solution of (13) in a given $Q =]s - r^2, s[\times B$ we simply denote by $\|a_i\|$ and $\|b_i\|$ the norms of the a_i's and b_i's in their respective spaces $L^{p,q}(Q)$. We also set

$$N = N_Q = \|a_2\| + \|a_4\| + \|b_2\|$$

which corresponds to the nonhomogeneous terms in (13).

Theorem 5.6. (i) *Any nonnegative subsolution u of* (13) *in $Q =]s - r^2, s[\times B$ satisfies*

$$\sup_{Q_\delta}\{u\} \leq C' e^{C\sqrt{K}r}((r^2 V)^{-1}\|u\|_{2,Q} + r^\theta N)$$

for $0 < \delta < 1$, where C depends only on n, but C' depends on n, α, μ, δ, θ, $\|a_1\|$, $\|a_3\|$, $\|a_5\|$, $\|b_1\|$, and r. Moreover, we have

$$\|\nabla u\|_{2,Q_\delta} \leq C' e^{C\sqrt{K}r} r^{-1}(\|u\|_{2,Q} + r^{1+\theta} V^{1/2} N).$$

(ii) *Any nonnegative solution u of* (13) *in Q satisfies*

$$\sup_{Q_-}\{u\} \leq C' e^{C\sqrt{K}r} \inf_{Q_+}\{u + r^\theta N\},$$

where C and C' are as above, but C' depends also on the parameters ε and η which enter the definition of Q_- and Q_+.

Note that Hölder regularity estimates can be obtained and that the other results given in [5] can also be adapted to our setting.

6. Fundamental solutions

The preceding results can be applied to the study of fundamental solutions of (12). More precisely, let $q_\Omega(t, y, t', y')$, $t' < t$, $y, y' \in \Omega$, be the minimal fundamental solution of (12) in $\mathbb{R} \times \Omega$, where Ω is an open set in M. It is the only nonnegative continuous function such that, for any $f \geq 0$ in $\mathscr{C}_0^\infty(\Omega)$, the function

$$\mathbf{Q}_{\Omega, t'} f(t, y) = \int q_\Omega(t, y, t', y') f(y') m(y')\, dv(y')$$

is a solution of (12) in $]t', +\infty[\times\Omega$ which tends to $f(y)$ as t tends to t' and satisfies $\mathbf{Q}_{\Omega, t'} f \leq u$ in $]t', t''[$ for any nonnegative solution u of (12) in $[t', t''[\times\Omega$ such that $f \leq u(t', \cdot)$ in Ω. For $y, y' \in M$ and $t, C, r > 0$ we set

$$E(C, t, y, y') = V(y, \sqrt{t})^{-1/2} V(y', \sqrt{t})^{-1/2} \exp(-C\rho^2(y, y')/t).$$

In this section, we prove some Gaussian estimates for the above fundamental solutions. Typically, these estimates involve quantities like E. We start with our most general result.

Theorem 6.1. *Let $0 < \delta < 1$ be fixed and suppose that $B \subset \Omega$. Then for all $y, y' \in \delta B$ and $t' < t < t' + r^2$ we have*

$$\begin{aligned} e^{-C_1(1+(\beta+K)\tau)} E(C_2, \tau, y, y') &\leq q_\Omega(t, y, t', y') \\ &\leq e^{C_3(1+\beta\tau+\sqrt{K\tau})} E(C_4^{-1}, \tau, y, y'), \end{aligned}$$

where $\tau = t - t'$. Moreover, for all $B_i = B(y_i, r_i) \subset M$, $i = 1, 2$, and all $t' < t$, we have

$$q_\Omega(t, y_1, t', y_2) \leq V_1^{-1/2} V_2^{-1/2} \exp(C_5(1+\beta\tau+\sqrt{K_1\tau_1}+\sqrt{K_2\tau_2})-\rho^2/C_6\tau),$$

where $\tau = t-t'$, $\tau_i = \inf\{\tau, r_i^2\}$, $V_i = V(y_i, \sqrt{\tau_i})$, $\rho = \rho(y_1, y_2)$, and the constants C_5 and C_6 depend only on n, α, and μ, whereas $C_1, \cdots, C_4$ depend also on δ.

Remark that, by applying Corollary 5.5, the above can be complemented with Hölder regularity estimates. The lower bound is a straightforward consequence of the Harnack inequality of Theorem 5.3. The upper bound is more involved. It follows from adapting the arguments of the proof of Theorem 6.3 below. When (M, g) has nonnegative Ricci curvature, and $\mathscr{L} = L$ has no lower order terms (i.e., $\beta = 0$), we obtain the following global two-sided estimate of q_M in terms of E.

Corollary 6.2. *Suppose that* (M, g) *has nonnegative Ricci curvature and that* $\beta = 0$. *Then, for all* $t' < t$, $\tau = t - t'$, *and all* $y, y' \in M$, *we have*

$$C_1^{-1} E(C_2, \tau, y, y') \le q_M(t, y, t', y') \le C_3 E(C_4^{-1}, \tau, y, y'),$$

where the constants $C_1, \cdots, C_4$ *depend only on* n, α, *and* μ.

Note that under the hypotheses of this corollary, and if $\mathscr{A}$ is independent of t, the above implies the two-sided estimate

$$C^{-1} \int_{\rho^2}^{+\infty} V(x, \sqrt{t})^{-1}\, dt \le G_L(x, y) \le C \int_{\rho^2}^{+\infty} V(x, \sqrt{t})^{-1}\, dt$$

for the Green function G_L of L on M (when it exists) (see [28]).

We pause here to discuss a typical probabilistic application of the Gaussian upper bound given by Theorem 6.1. For the purpose of this illustration, assume that there exists $K \ge 0$ such that $\mathrm{Ric} \ge -Kg$ on M. Also assume that the coefficients of L are time independent, and that $\mathscr{X} = 0$ and $b = 0$. Under these hypotheses, we can consider the diffusion process X_t governed by $\mathscr{L}$, and the corresponding family of probability measures $\{P_x, x \in M\}$ on the set of continuous paths in M. Of course, we have

$$P_x(X_t \in U) = \int_U q_M(t, x, 0, y) m(y)\, dv(y)$$

(it will be shown in the next section that q_M has total mass one under the above hypotheses). Fix $x \in M$ and $r > 0$, and consider the first time $\tau(x, r)$ at which the process X_t, starting at x at $t = 0$, exists from the ball $B(x, r)$. With this notation, we have

$$P_x(\tau(x, r) \le t) \le \exp(C\sqrt{K}(r + \sqrt{t}) - r^2/C't), \qquad x \in M, r, t > 0.$$

To obtain this result we can argue as in [36, Lemma 3] and reduce the proof to the estimate

$$\int_{\{\rho(x, y) > r\}} q_M(t, x, 0, y) m(y)\, dv(y) \le \exp(C\sqrt{K}(r + \sqrt{t}) - r^2/C't),$$

which is a consequence of the Gaussian upper bound of Theorem 6.1 and of the volume estimate (3).

The Gaussian upper bounds given by Theorem 6.1 can be improved under some additional hypotheses. Namely, let us assume that $\mathscr{L} = L$

has no lower order terms, and that $\mathscr{A} \in \mathrm{Sym}(T_M)$ and is independent of t. In this case, we consider the minimal symmetric submarkovian semigroup e^{-tL} associated with the Dirichlet form

$$\mathscr{D}_{\mathscr{A},m}(f) = \int g(\mathscr{A}\nabla f, \nabla f) m\, dv, \qquad f \in \mathscr{C}_0^\infty(\Omega).$$

We fix Ω, and denote by $q_t(y, y')$ the kernel of e^{-tL} on $L^2(\Omega, m\,dv)$. Of course, with our previous notation, we have $q_t(y, y') = q_\Omega(t, y, 0, y')$. We consider the distance $\tilde{\rho}$ induced by the metric

$$\tilde{g}(X, Y) = g(\mathscr{A}^{-1}X, Y).$$

Also denote by

$$\nu = \inf\{(Lf, fm)/(f, fm), f \in \mathscr{C}_0^\infty(\Omega)\}$$

the bottom of the spectrum of L on $L^2(\Omega, m\,dv)$. Our next result improves upon the upper bound given by Theorem 6.1 by taking into account the roles of ν and $\tilde{\rho}$. Moreover, using the L^2 analyticity of e^{-tL}, we extend our estimate to time derivatives of the kernel.

Theorem 6.3. *Let $k \in \mathbb{N}$ be fixed. Then there exist a constant C depending only on n, and a constant C_k depending only on n, α, μ, and k, such that, for all $t > 0$ and $B_i = B(y_i, r_i) \subset \Omega$, $i = 1, 2$, we have*

$$|\partial_t^k q_t(y_1, y_2)| \le C_k(1 + \nu(t_1 + t_2))^{1+n/2} e^{C(\sqrt{K_1 t_1} + \sqrt{K_2 t_2})} (V_1 V_2)^{-1/2} \times t^{-k}(1 + \tilde{\rho}^2/t)^{n/2}(1 + \nu t + \tilde{\rho}^2/t)^k \exp(-\nu t - \rho^2/4t),$$

where $t_i = \inf\{t, r_i^2\}$, $V_i = V(y_i, \sqrt{t_i})$, and $\rho = \rho(y_1, y_2)$.

Once more, Hölder regularity estimates for $\partial_t^k q_t$ can be deduced from Corollary 5.5. Remark also that, when $k = 0$, the restriction $B_i \subset \Omega$ can be removed since q_Ω is an increasing function of Ω. In another direction, we note that the above also holds for the kernel corresponding to Neumann boundary conditions on Ω. In this case, the condition $B_i \subset \Omega$ is essential even when $k = 0$. The above proof uses an idea of Davies [12] (which has been used extensively by many authors) together with the Harnack-Moser inequality of Theorem 5.1. This technique is a variation on an approach used by Varopoulos in [41].

Proof. Fix $\lambda \in \mathbb{R}$ and a bounded function ψ satisfying $|\tilde{\nabla}\psi|^2 = g(\mathscr{A}\nabla\psi, \nabla\psi) \le 1$. For any nice (complex) function f, we set $f_z(y) = e^{\lambda\psi(y)} e^{-Lz}(e^{-\lambda\psi} f)(y)$ for $z = se^{i\theta} \in \mathbb{C}$, $s > 0$, $|\theta| \le \frac{1}{2}\varepsilon$, where $0 \le \varepsilon \ll 1$ is a small fixed parameter. We have

$$\begin{aligned}\partial_s \|f_z\|_2^2 &= -2\,\mathrm{Re}\left(e^{i\theta}\int g(\mathscr{A}\nabla(e^{-\lambda\psi}f_z), \nabla(e^{\lambda\psi}\overline{f}_z))m\,dv\right)\\ &= -2\,\mathrm{Re}\left(e^{i\theta}\int\{g(\mathscr{A}\nabla f_z, \nabla\overline{f}_z) - \lambda^2 g(\mathscr{A}\psi, \nabla\psi)|f_z|^2\right.\\ &\qquad\left. + \lambda(f_z g(\mathscr{A}\nabla\psi, \nabla\overline{f}_z) - \overline{f}_z g(\mathscr{A}\nabla f_z, \nabla\psi))\}m\,dv\right)\\ &\le -2(\cos\theta - \sin\theta)\int g(\mathscr{A}\nabla f_z, \nabla\overline{f}_z)m\,dv\\ &\quad + 2\lambda^2(\cos\theta + \sin\theta)\int |f_z|^2 m\,dv\\ &\le 2\{-\nu(1-\varepsilon) + \lambda^2(1+\varepsilon)\}\|f_z\|_2^2.\end{aligned}$$

From this, we deduce

$$\|f_z\|_2^2 \le e^{2(\lambda^2(1+\varepsilon)-\nu(1-\varepsilon))t}\|f\|_2^2.$$

Applying the Cauchy formula to f_z (with a circle of radius $\varepsilon t/10$ and center $t > 0$), we obtain

$$\|\partial_t^k f_t\|_2 \le C_0^k k!(\varepsilon t)^{-k} e^{(\lambda^2(1+\varepsilon)-\nu(1-\varepsilon))t}\|f\|_2^2.$$

We now introduce the function

$$u(t, y) = e^{-\lambda\psi(y)}e^{\nu(1-\varepsilon)t}\partial_t^k f_t(y) = e^{\nu(1-\varepsilon)}\partial_t^k e^{-tL}(e^{-\lambda\psi}f)(y),$$

where $f \in L^2$ is now a real function. The function u is a solution of $(\partial_t + L - \nu(1-\varepsilon))u = 0$, and we apply Theorem 5.1 on $]t-\eta t_1, t[\times B(y_1, \sqrt{\eta t_1})$, where $t_i = \inf\{t, r_i^2\}$ and $0 < \eta \ll 1$. This yields

$$|u(t, y_1)|^2 \le (1+\nu\eta t_1)^{1+n/2} e^{C(1+\sqrt{K_1\eta t_1})}(\eta t_1 V(y_1, \sqrt{\eta t_1}))^{-1} \times \int_{t-\eta t_1}^{t}\int_{B(y_1, \sqrt{\eta t_1})} u^2.$$

Introducing $e^{\lambda\psi(y)}$, we get

$$e^{2\lambda\psi(y_1)}|u(t, y_1)|^2 \le A(k, t, \varepsilon)^2 W_1(\eta)^2 e^{2|\lambda|\sqrt{\eta t_1}+2\lambda^2(1+\varepsilon)t}\|f\|_2^2,$$

where we have set $A(k, t, \varepsilon) = C_0^k k!(\varepsilon t)^{-k}$, and

$$W_1(\eta) = (1+\nu\eta t_1)^{(2+n)/4} e^{C(1+\sqrt{K_1\eta t_1})/2} V(y_1, \sqrt{\eta t_1})^{-1/2}.$$

Taking the supremum over all $f \in L^2(B(y_2, \sqrt{\eta t_2})$ with $\|f\|_2 = 1$, we obtain

$$e^{2\lambda(\psi(y_1)-\psi(y_2))} \int_{B(y_2, \sqrt{\eta t_2})} |e^{2\nu(1-\varepsilon)t} \partial_t^k q_\Omega(t, y_1, z)|^2 \, dv(z)$$
$$\leq A(k, t, \varepsilon)^2 W_1(\eta)^2 e^{2|\lambda|\sqrt{\eta}(\sqrt{t_1}+\sqrt{t_2})+2\lambda^2(1+\varepsilon)t}.$$

Again applying Theorem 5.1, this time on $]t - \eta t_2, t[\times B(y_2, \sqrt{\eta t_2})$, we get

$$|\partial_t^k q_t(y_1, y_2)|$$
$$\leq A(k, t, \varepsilon) W_1(\eta) W_2(\eta) e^{|\lambda|\sqrt{\eta}(\sqrt{t_1}+\sqrt{t_2})+\lambda^2(1+\varepsilon)t-\lambda(\psi(y_1)-\psi(y_2))-\nu(1-\varepsilon)t}.$$

We now choose $\lambda = \tilde{\rho}(y_1, y_2)/2(1+\varepsilon)t = \tilde{\rho}/4(1+\varepsilon)t$, and ψ such that $\psi(y_1) - \psi(y_2) = \tilde{\rho}$, which is compatible with the condition $|\tilde{\nabla}\psi| \leq 1$. Then we have

$$|\partial_t^k q_t(y_1, y_2)| \leq A(k, t, \varepsilon) W_1(\eta) W_2(\eta) e^{2\tilde{\rho}\sqrt{\eta/t}-\tilde{\rho}^2/4(1+\varepsilon)t-\nu(1-\varepsilon)t}.$$

We choose $\eta = (10(1+\tilde{\rho}^2/t))^{-1}$, and get

$$|\partial_t^k q_t(y_1, y_2)| \leq A(k, t, \varepsilon) W(B_1, \tilde{\rho}, t) W(B_2, \tilde{\rho}, t) e^{-\tilde{\rho}^2/4(1+\varepsilon)t-\nu(1-\varepsilon)t},$$

where we have set

$$W(B, R, t) = (1+\nu\tau)^{(2+n)/4} e^{C(1+\sqrt{K\tau})} V(x, \sqrt{\tau/(1+R^2/t)})^{-1/2}$$

for $B = B(x, r)$, $R > 1$, and $\tau = \inf\{t, r^2\}$. If $k = 0$, we can take $\varepsilon = 0$, and obtain

$$q_t(y_1, y_2) \leq W(B_1, \tilde{\rho}, t) W(B_2, \tilde{\rho}, t) \exp(-\nu t - \tilde{\rho}^2/4t)$$
$$\leq (1+\nu(t_1+t_2))^{1+n/2} e^{C(1+\sqrt{K_1 t_1}+\sqrt{K_2 t_2})} (V_1 V_2)^{-1/2}$$
$$\times (1+\tilde{\rho}^2/t)^{n/2} \exp(-\nu t - \tilde{\rho}^2/4t).$$

Also, if $\nu = 0$, we take $\varepsilon = (10(1+\tilde{\rho}^2/t))^{-1}$ and get

$$|\partial_t^k q_t(y_1, y_2)| \leq C_0^k k! W(B_1, \tilde{\rho}, t) W(B_2, \tilde{\rho}, t) t^{-k} (1+\tilde{\rho}^2/t)^k \exp(-\tilde{\rho}^2/4t).$$

In any case, we end the proof of Theorem 6.3 by taking $\varepsilon = (1+\nu t+\tilde{\rho}^2/t)^{-1}$. It is worth specializing this result to the case of non-negative Ricci curvature.

Corollary 6.4. *Assume that $\Omega = M$ and that g has nonnegative Ricci curvature. Then we have*

$$|\partial_t^k q_t(x, y)| \le C_k V\left(x, \sqrt{t/(1+\tilde{\rho}^2/t)}\right)^{-1/2} V\left(y, \sqrt{t/(1+\tilde{\rho}^2/t)}\right)^{-1/2} \times t^{-k}(1+\tilde{\rho}^2/t)^k \exp(-\tilde{\rho}^2/4t)$$
$$\le C_k V(x, \sqrt{t})^{-1/2} V(y, \sqrt{t})^{-1/2} (1+\tilde{\rho}^2/t)^{k+n/2} t^{-k} \exp\left(-\frac{\tilde{\rho}^2}{4t}\right),$$

where C_k depends on n, α, μ, and k.

The estimates given in Theorem 6.3 and Corollary 6.4 have to be compared to the upper bounds given in [28], [13], [14], [15], and [41] for the case where $L = \Delta$. Even in this case, the above has only improved slightly upon previously known results.

As far as space regularity is concerned, it is well known that Hölder regularity is the best that can be expected without quantitative smoothness assumptions on $\mathscr{A}$ and m. When $L = \Delta$ however, the Li-Yau gradient estimate gives a bound on the first space derivatives of the heat kernel. Indeed, let $\Omega \subset M$ be a fixed open set, and denote by $h_t = h_{\Omega, t}$ the heat kernel associated with Δ and the Dirichlet boundary condition on Ω. If $B \subset \Omega$, we have on $\frac{1}{2}B$

$$|\nabla h_t|^2 \le C(t^{-1} + r^{-2} + K)h_t^2 + |\partial_t h_t| h_t$$

(see [28]), and a Gaussian estimate for the gradient of h_t follows readily from the preceding results. The best known estimates for higher space derivatives of the kernel h_t were obtained in [9]. The authors of that paper remark that, if $|D^k f|$ is the norm of the kth covariant derivative of f, then $|D^k h_t|$ satisfies a parabolic inequality. Using Theorems 5.6 and 6.3 and the ideas of [9], we obtain the following result which improves upon [9, Theorem 7, p. 1059]. Let $B_i = B(x_i, r_i)$, $i = 1, 2$, be two balls in M. Recall that K_i is the smallest nonnegative number such that $\mathrm{Ric} \ge -K_i g$ on $2B_i$. We define $K_{j,i}$, $j \ge 0$, $i = 1, 2$, to be a bound on the jth covariant derivatives of the curvature tensor on the ball $2B_i$.

Theorem 6.5. *Let $k_1, k_2, l \in \mathbb{N}$ be fixed, and assume that $B_i = B(y_i, r_i) \subset \Omega$, $i = 1, 2$. Then there exists C depending on n, ν_0, k_1, k_2, l, r_1, r_2, K_1, K_2, $K_{i,1}$ with $0 \le i \le k_1 - 1$, and $K_{j,2}$ with $0 \le j \le k_2 - 1$ such that for all $t > 0$ we have*

$$|D_1^{k-1} D_2^{k_2} \partial_t^l h_t(y_1, y_2)| \le C V_1^{-1/2} V_2^{-1/2} t_1^{-k_1/2} t_2^{-k_2/2} t^{-l} (1+\rho^2/t)^{(k_1+k_2+n)/2} \times (1 + \nu_0 t + \rho^2/t)^l \exp(-\nu_0 t - \rho^2/4t),$$

where $t_i = \inf\{t, r_i^2\}$, $V_i = V(y_i, \sqrt{t_i})$, $\rho = \rho(y_1, y_2)$, *and*

$$\nu_0 = \inf\{(\Delta f, f)/\|f\|_2^2, f \in \mathscr{C}_0^\infty(\Omega)\}.$$

Moreover, when $k_i = 0, 1$ *we can replace* C *by*

$$C_k(1 + \nu_0(t_1 + t_2))^{1+n/2} \exp\{C'(\sqrt{K_1 t_1} + \sqrt{K_2 t_2})\},$$

where C' *depends only on* n, *but* C_k *depends on* n *and* k.

The same technique yields similar results for the kernels corresponding to Neumann boundary conditions.

7. Uniqueness results and Liouville type theorems

Uniqueness questions for the heat equation $(\partial_t + \Delta)u = 0$ on complete Riemannian manifolds are by now quite well understood. Uniqueness on L^p with $1 < p < +\infty$ holds without any curvature assumption (see [38] and [24]). Uniqueness for the positive or L^1 Cauchy problem holds under the assumption that the Ricci curvature has a negative quadratic lower bound. By this we mean that there exists $x_0 \in M$ and $C \geq 0$ such that $\mathrm{Ric}_x \geq -C_0(1 + \rho(x_0, x))^2 g_x$ for all $x \in M$ (see [28] and [24], and also [6] for a counterexample showing the sharpness of the curvature condition). Using the Harnack-Moser inequalities of §5 we are going to show that these results hold as well for equation (12). We start with a very general result that may be known since its proof follows the same line as its classical Euclidean counterpart.

Theorem 7.1. *Assume that* u *is a nonnegative subsolution of* (12) *on* $]s, s+T[\times M$ *which satisfies*

$$\int_s^{s+T} \int e^{-2\gamma\rho^2} |u(t,x)|^2 \, dt \, dv < +\infty \quad \text{and} \quad \lim_{t \to s} \|e^{-\gamma\rho^2} u(t, \cdot)\|_2 = 0$$

for some $\gamma \geq 0$, *where* $\rho(x) = \rho(x_0, x)$ *for some fixed* x_0. *Then,* $u(t,x) = 0$ *for all* $(t,x) \in]s, s+T[\times M$.

We refer the reader to [3] and [4] where earlier references are also given. It is worth noticing with [3] that if $\mathscr{L}$ has no lower order terms and $\mathscr{A} \in \mathrm{Sym}(T_M)$, then the above holds even without the ellipticity hypothesis $g(\mathscr{A}X, X) \geq \alpha^{-1}|X|^2$. What is needed is only $\mathscr{A}$ nonnegative and bounded. Using Theorems 7.1, 5.1, and 5.3, as well as (3), we easily obtain

Theorem 7.2. *Suppose that the Ricci curvature has a negative quadratic lower bound on* M.

(i) *If* u *is a nonnegative solution of* (12) *in* $]s, s+T[\times M$ *such that* $\lim_{t\to s} u(x, t) = 0$, *then* $u = 0$ *in* $]s, s+T[\times M$.

(ii) *Let* $0 < p \leq 1$. *If* u *is a nonnegative subsolution of* (12) *in* $]s, s+T[\times M$ *such that*

$$\int_s^{s+T} \int |u(t, x)|^p \, dt \, dv < +\infty$$

and $\lim_{t\to s} \|u(t, \cdot)\|_p = 0$, *then* $u = 0$ *in* $]s, s+T[\times M$.

From the above theorem, and the general Widder's theorem in [2], we obtain

Theorem 7.3. *Suppose that the Ricci curvature of* g *has a negative quadratic lower bound on* M. *Let* u *be a nonnegative solution of* (12) *in* $]s, s+T[\times M$. *Then there exists a unique nonnegative Borel measure* ξ *such that* $u = \mathbf{Q}_{M,s}\xi$.

We refer the reader to [2] for results on uniqueness of isolated singularities. Other results in [4] can be adapted to our setting. As an application of Theorem 7.2, we also obtain that, under the current curvature hypothesis, and if $\mathscr{X} = 0$ and $b = 0$, then $\int_M q_M(t, x, s, y)\, dv(y) = 1$ for all $t > s$ and $x \in M$. When the coefficients of $\mathscr{L}$ are independent of t and the Ricci curvature is bounded below on M, we can apply the method of [22] to obtain another proof of the uniqueness of the nonnegative Cauchy problem. Moreover, in this case we obtain that the minimal nonnegative solutions u of (12) in $]-\infty, T[\times M$ are of the form $u = e^{\lambda t} w$, where w is a minimal nonnegative solution of $\mathscr{L}w = \lambda w$ (see [22]).

Another question of interest is whether or not $\mathbf{Q}_{M,s}(\mathscr{C}_0(M)) \subseteq \mathscr{C}_0(M)$, where $\mathscr{C}_0(M)$ is the class of continuous functions that tend to 0 at infinity (see [45], [6], [25]). It follows from Corollary 5.5, Theorem 6.1, and the volume estimate (3) that we have $\mathbf{Q}_{M,s}(\mathscr{C}_0(M)) \subseteq \mathscr{C}_0(M)$, when the Ricci curvature of g admits a negative quadratic lower bound on M.

We now pass to Liouville type properties. We consider an operator L defined by (8) (i.e., with no lower order terms), where $\mathscr{A}$ is independent of t, and m and $\mathscr{A}$ satisfy the ellipticity hypotheses (10) and (9). We say that u is L-harmonic when $Lu = 0$. An immediate consequence of Theorem 5.3 and Corollary 5.4 is the following generalization of a theorem of Yau (see [43]).

Theorem 7.4. *Assume that* (M, g) *has nonnegative Ricci curvature. Then, any L-harmonic function which is bounded below is constant. Moreover, there exists a constant* γ, $0 < \gamma \leq 1$, *depending only on* n, α, *and* μ, *such that any L-harmonic function satisfying* $\lim_{r\to 0}(r^{-\gamma} \sup_{B(x_0, r)}\{|u|\}) = 0$ *for some fixed* x_0, *is constant.*

In [30], Lyons gave examples showing that, in general, Liouville's properties are not stable under quasi-isometric changes of metrics. However, the above theorem, as well as the rest of this paper, shows that many results are stable under such changes of metric when we start out with a lower bound hypothesis on the Ricci curvature.

We conclude this section by noting that the work [26] of Li and Schoen, where they study the triviality of the nonnegative L-subharmonic functions which belong to L^p for some $0 < p < +\infty$, can be extended to the present setting. For instance, following [24] and [26], one can show that any nonnegative L-subharmonic function that belongs to L^1 is constant under the hypothesis that the Ricci curvature of g admits a negative quadratic lower bound. In [17], it is shown that the space of bounded (Δ-) harmonic functions is of finite dimension when (M, g) has nonnegative Ricci curvature outside a compact set. It seems safe to conjecture that this result is also stable and holds for L-harmonic functions. Moreover, the dimension of the space of bounded L-harmonic functions should not depend on L. However, we have not been able to answer these questions.

8. Functions of the Laplace operator

In [8] and [39] the finite propagation speed for the wave equation $(\partial_t^2 + \Delta)u = 0$ is used to obtain estimates on the kernel of some functions of the Laplace operator. One of the other tools used in these papers is Moser's iteration. Following the arguments in [8] and [39], and using Theorem 5.1, we obtain some new results that improve upon or complement those of [8] and [39].

Consider an operator $L = L_{\mathscr{A}, m}$ as in (8), where $\mathscr{A}$ and m satisfy the usual uniform ellipticity hypotheses (9) and (10). Moreover, assume that $\mathscr{A} \in \mathrm{Sym}(T_M)$ and is independent of t. Denote by $\widetilde{B}(x, r)$ the ball corresponding to the distance $\tilde{\rho}$ associated with the metric $g(\mathscr{A}^{-1}\cdot, \cdot)$ (see §4). Also, let $\nu = \inf\{(Lu, um)/(u, um),\ u \in \mathscr{C}_0^\infty(M)\}$ be the bottom of the spectrum of L on $L^2(M, m\,dv)$. All the L^p-norms in this section are taken with respect to the measure $m\,dv$ (note that it is of very little importance since m and m^{-1} are bounded). Given a bounded real function, denote by k_f the kernel of the operator $F = f(\sqrt{L - \nu})$ which is defined on $L^2(M, m\,dv)$ by spectral theory. When f is even, we can write

$$F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(s) \cos(s\sqrt{L - \nu})\, ds,$$

where $\hat{f}$ is the Fourier transform of f. Hence, using the finite propagation speed property of the equation $(\partial_t^2 + L)u = 0$, for any function u with support in $\Omega_1 \subset M$, and any $\Omega_2 \subset M$ such that $\tilde{\rho}(\Omega_1, \Omega_2) = R$ (see [8, Corollary 1.2]), we have

$$\|(L-\nu)^k F(L-\nu)^l u\|_{2,\Omega_2} \leq \frac{\sqrt{2}}{\sqrt{\pi}} \|u\|_{2,\Omega_1} \int_R^\infty |\hat{f}^{(2k+2l)}(s)|\, ds.$$

Following [8] and [39], we now introduce the class $\mathscr{F}(\varphi, b, \beta)$ of even functions f such that

$$|\hat{f}^{(k)}(s)| \leq C(k/b)^k \varphi(|s|), \qquad k \geq 0, |s| \geq \beta,$$

where $b \in]0, +\infty[$, $\beta \in [0, +\infty[$, and the integrable function $\varphi: [0, +\infty[\to [0, +\infty[$ are fixed. We set $\psi(r) = \int_r^\infty \varphi(s)\, ds$. Arguing as in [8], we obtain (compare to §2 and Theorem 3.1 of [8])

Theorem 8.1. *Fix $0 < \delta < 1$ and $f \in \mathscr{F}(\varphi, b, \beta)$. Let $B_i = B(x_i, r_i)$, $i = 1, 2$, be two balls with $r_i \leq \alpha^{-1} b$ and $R = \tilde{\rho}(\widetilde{B}_1, \widetilde{B}_2) \geq \beta$. Then the kernel k_f satisfies*

$$|k_f(y_1, y_2)| \leq C(K_1, \nu, r_1) C(K_2, \nu, r_2) V_1^{-1/2} V_2^{-1/2} \psi(R), \qquad y_i \in \delta \widetilde{B}_i.$$

Moreover, there exists γ, $0 < \gamma < 1$, such that

$$|k_f(z_1, z_2) - k_f(y_1, y_2)| \leq \overline{\rho}^\gamma e^{C(1+(K_1+\nu)r_1^2 + (K_2+\nu)r_2^2)} V_1^{-1/2} V_2^{-1/2} \psi(R),$$

where $\overline{\rho} = \max\{\rho(y_1, z_1)/r_1, \rho(y_2, z_2)/r_2\}$, $y_i, z_i \in \delta \widetilde{B}_i$.

For instance, the above theorem yields rather sharp estimates on the kernels $q_{\sigma,t}$ of the semigroup $Q_{\sigma,t} = e^{-tL^\sigma}$, where $\sigma = 1, 2, \cdots$ is a fixed integer. Indeed, set $f_\sigma(\lambda) = \exp(-\lambda^{2\sigma})$, $\lambda \in \mathbb{R}$. Then, there exist two positive constants a and b such that $f_\sigma \in \mathscr{F}(\varphi_\sigma, b, 0)$, where $\varphi_\sigma(s) = \exp(-as^{2\sigma/(2\sigma-1)})$ (see [20, Chapter 4]). Hence, we obtain

Corollary 8.2. *Assume that (M, g) has nonnegative Ricci curvature, and fix an integer $\sigma = 1, 2, \cdots$. Then the kernel $q_{\sigma,t}$ of e^{-tL^σ} satisfies*

$$|\partial_t^k q_{\sigma,t}(x, y)| \leq C_k' V(x, t^{\frac{1}{2\sigma}})^{-\frac{1}{2}} V(y, t^{\frac{1}{2\sigma}})^{-\frac{1}{2}} t^{-k} \exp\left(-\left(\frac{\rho}{C_k t^{\frac{1}{2\sigma}}}\right)^{\frac{2\sigma}{(2\sigma-1)}}\right)$$

for all $x, y \in M$, $t > 0$, and any fixed $k = 0, 1, \cdots$, where, C_k and C_k' are constants which depend only on n, α, μ, σ, and k. In particular, we have $\|\partial_t^k q_{\sigma,t}(x, \cdot)\|_1 \leq Ct^{-k}$ for $t > 0$ and $x \in M$, which shows that the semigroup e^{-tL^σ} is a bounded analytic semigroup on L^p for all $p \in [1, +\infty[$.

We now turn to some results concerning the L^p-continuity of F and the L^p-spectrum of L under a global curvature hypotheses on M. Namely, we assume that $-K$ is a lower bound for the Ricci curvature of g on M $(K \geq 0)$. Moreover, we introduce the parameter λ, associated with $\mathscr{A}$, and defined as the smallest nonnegative number such that

$$(14) \qquad V_{\mathscr{A}}(x, r)/V_{\mathscr{A}}(x, r') \leq C(r/r')^{\gamma} e^{\lambda r}, \qquad 0 < r' < r, x \in M,$$

for some $C, \gamma > 0$, where $V_{\mathscr{A}}(x, r) = v(\widetilde{B}(x, r))$. The parameter λ plays an important role in what follows. Remark that, if we denote λ_0 to be the parameter associated with $\mathscr{A} = \mathrm{Id}$, the ellipticity of $\mathscr{A}$ implies $\alpha^{-1}\lambda_0 \leq \lambda \leq \alpha\lambda_0$. Moreover, the volume estimate (2) shows that $\lambda_0 \leq \sqrt{(n-1)K}$. In other words, (14) is always satisfied for some $\lambda \leq \alpha\sqrt{(n-1)K}$. We also notice, as in [39], that λ is related to the bottom of the spectrum of L by the inequality $\nu \leq \lambda^2/4$ which is obtained by considering the test functions $e^{-\beta\tilde{\rho}}$ with $\beta \nearrow \lambda/2$. Following [39], we introduce the classes of functions $\mathscr{S}_W^{\#} = \bigcup_{b>0} \mathscr{F}(e^{-Ws}, b, 0)$ and $\mathscr{S}_W^{\#\#} \subset \mathscr{S}_W^{\#}$ which is the set of even smooth functions f on $\{|\operatorname{Im}(z)| \leq W\} \subset \mathbb{C}$, holomorphic in $\{|\operatorname{Im}(z) < W\}$, and satisfying $|f(z)| \leq ce^{-b|z|}$ on $\{|\operatorname{Im}(z)| \leq W\}$ for some $c, b > 0$ (see [39, Lemma 5.5]).

Theorem 8.3. *Suppose that the Ricci curvature is bounded below on* (M, g).

(i) *If* $f \in \mathscr{S}_W^{\#}$ *with* $W > \lambda/2$, *then* $F = f(\sqrt{L - \nu})$ *is bounded on* L^p *for* $1 \leq p \leq +\infty$.

(ii) *If* $f \in \mathscr{S}_W^{\#\#}$, *then* F *is bounded on* L^p *provided that* $W > |1/p - 1/2|\lambda$ *and* $1 < p < +\infty$.

Corollary 8.4. *Suppose that the Ricci curvature is bounded below on* (M, g). *Then, the* L^p*-spectrum of* L *satisfies*

$$\operatorname{spec}_p(L) \subset \{y = \nu - z^2, |\operatorname{Re}(z)| \leq 1/p - 1/2|\lambda\} \cap \{\operatorname{Re}(y) \geq 4\nu(p-1)p^{-2}\}.$$

In particular, if (M, g) *has subexponentional volume growth and satisfies* $\inf_M\{V(x, 1)\} > 0$, *or if* (M, g) *has nonnegative Ricci curvature, then* $\operatorname{spec}_p(L) \subset [0, +\infty[$.

Part (ii) of the above theorem follows from (i) by complex interpolation (see [39, Proposition 5.6]). To prove (i) we show that the kernel k_f satisfies $\|k_f(x, \cdot)\|_1 = \|k_f(\cdot, x)\|_1 \leq C$. This can be deduced from the volume hypothesis (14), Theorem 5.1, and the argument in [39, p. 777], and the corollary can readily follow (see Lemma 5.7 and Theorem 5.8 in [39]). We refer the reader to [16] and [39] and the references given there for examples involving hyperbolic spaces and their quotients which show the sharpness of Corollary 8.4.

9. Poincaré inequalities

In this section, we discuss the proof of Theorem 3.2 as well as some related results. We start by recalling the following simple result which is proved in [26].

Theorem 9.1. *Given $1 \le p < +\infty$, there exists C, depending only on n and p, such that for any ball B satisfying $B \neq M$, we have*

$$\|f\|_p \le e^{C(1+\sqrt{K}r)} r \|\nabla f\|_p, \qquad f \in \mathscr{C}_0^\infty(\tfrac{1}{2}B).$$

When $p = 2$, the above can be viewed as a lower bound for the first eigenvalue of the Laplacian with Dirichlet boundary condition in the ball B.

We now describe some of the results obtained by Buser in [7], which yield a lower bound for the first nonzero eigenvalue of the Laplacian with Neumann boundary condition instead. Consider the Cheeger constant $h(B) = \inf\{v_{n-1}(\partial\Omega \cap B)/v(\Omega)\}$, where Ω runs over the open subset of B such that $v(\Omega) \le v(B)/2$. It is proved in [7] that there exists a constant C, depending only of n, such that, for all $B \in M$,

$$h(B) \ge r e^{-C(1+\sqrt{K}r)}. \tag{15}$$

The classical Cheeger inequality can be rephrased, in this setting, as

$$\int_B |f - f_B|^2 \, dv \le 4h(B)^{-2} \int_B |\nabla f|^2 \, dv, \qquad f \in \mathscr{C}^\infty(B),$$

which, together with (15), gives (6). More generally, it is well known that

$$\int_B |f - f_B|^p \, dv \le 2p^p h(B)^{-p} \int_B |\nabla f|^p \, dv$$

for all $1 \le p < +\infty$. A weaker version of (6) (which suffices for the purpose of this paper) can be obtained by using two-sided estimates on the heat kernel with Neumann boundary condition on balls and the idea of Kusuoka-Stroock [23, p. 435]. As explained before, we need to prove a weighted version of the Buser inequality. Fix a nonincreasing function $\varphi\colon [0, +\infty[\to [0, 1]$ such that $\varphi(t) = 0$ for $t > 1$, and which satisfies

$$\varphi(t + (1-t)/2) \ge \beta\varphi(t), \qquad 1/2 \le t \le 1,$$

for some $\beta > 0$. Also, we set $\Phi_B = \varphi(\rho(x, \cdot)/r)$ for $B = B(x, r)$. Examples of functions φ satisfying the above conditions are $\varphi = 1$ on

$[0, 1]$, or $\varphi = 1$ on $[0, \delta]$, $\varphi(t) = (1-t)/(1-\delta)$ on $]\delta, 1]$, as well as φ^γ, where φ satisfies the above conditions and $\gamma > 0$.

Theorem 9.2. *Given $1 \le p < +\infty$, there exist C, depending only on n, and $C_{p,\varphi}$, depending on n, p, and φ, such that, for any ball $B \in M$, we have*

$$\left(\int |f - f_\Phi|^p \Phi\, dv\right)^{1/p} \le C_{p,\varphi} e^{C\sqrt{K}r} r \left(\int |\nabla f|^p \Phi\, dv\right)^{1/p}, \quad f \in \mathscr{C}^\infty(B).$$

It turns out that Theorem 9.2 can be deduced from (15) by adapting an argument from [21] which uses a Witney covering of B. The details are spelled out in [36] in a different setting, and we refer the reader to these two papers for the proof.

We end this section with the discussion of yet another kind of Poincaré inequality. Although the result obtained below is not used at all in this paper, it can be of some independent interest. In view of the preceding, it is tempting to ask for which nice, nonnegative, integrable function ψ does C_ψ exist such that

$$\int |f - f_\psi|^2 \psi\, dv \le C_\psi \int |\nabla f|^2 \psi\, dv, \qquad f \in \mathscr{C}^\infty(M) \cap L^2(M, \psi\, dv), \tag{16}$$

where $f_\psi = \int f\psi\, dv / \int \psi\, dv$. One way to study this question is to set $u = f\psi^{1/2}$ and remark that

$$\int |\nabla f|^2 \psi\, dv = ((\Delta + b)u, u),$$

where $b = \frac{1}{4}|\nabla \ln \psi|^2 - \frac{1}{2}\Delta \ln \psi$. The transformation $Uf = u$ is an isometry from $L^2(M, \psi\, dv)$ to $L^2(M, dv)$. Hence, the symmetric, nonnegative operator $\Delta + b$ admits 0 as an eigenvalue with eigenfunction $\psi^{1/2}$. If we assume that the support of ψ is connected, 0 is a simple eigenvalue and our concern is now to prove that there is a gap in the spectrum of $\Delta + b$. A way to do this is to obtain a positive lower bound for the essential spectrum of $\delta + b$. By classical arguments (involving some local Sobolev inequalities), if we assume that b satisfies $b \in L^1_{\text{loc}}$ and $b_- = \max\{-b, 0\} \in L^p + L^\infty$ for some $p > n/2$, then

$$\text{ess spec}(\Delta + b) = \sup_N \{\inf\{((\Delta + b)u, u)/\|u\|_2^2, u \in \mathscr{C}_0^\infty(M \backslash N)\}\},$$

where the sup is taken over all compacts $N \subset M$ (see [1]).

We give some specific examples of interest. Let $\psi = e^{-\omega(\rho)}$, where ω is a smooth function from $[0, +\infty[$ to $\mathbb{R}$, and $\rho = \rho(x_0, \cdot)$ for some fixed $x_0 \in M$. In this case, we have

$$b = |\omega'(\rho)|^2/4 + (\omega''(\rho) + \omega'(\rho)\Delta\rho)/2.$$

If we define $K(\rho)$ to be such that $\mathrm{Ric} \geq -K(\rho)g$ on $B(x_0, \rho)$, and suppose that $\omega'(\rho)$ is nonnegative for ρ large enough, then by a well-known comparison inequality we obtain

$$b \geq |\omega'(\rho)|^2/4 + (\omega''(\rho) - (n-1)(\rho^{-1} + (\sqrt{K})\omega'(\rho))/2, \qquad \rho \gg 1.$$

Thus the above arguments lead to

Theorem 9.3. *Fix $x_0 \in M$ and suppose that there exist k, $\beta \geq 0$ such that*

$$\mathrm{Ric} \geq -k(1+\rho)^{2\beta} g$$

on M. Let $a > 0$ and $\gamma \geq 1$ be such that either $\gamma > \beta + 1$ and $a > 0$, or $\gamma = \beta + 1$ and $a > \sqrt{(n-1)k}$. Then, there exists a constant C, depending on (M, g), x_0, a, and γ, such that

$$\int |f - \overline{f}|^2 e^{-a\rho^\gamma}\, dv \leq C \int |\nabla f|^2 e^{-a\rho^\gamma}\, dv, \quad f \in \mathscr{C}^\infty(M) \cap L^2(M, e^{-a\rho^\gamma}\, dv),$$

where $\overline{f} = \int f e^{-a\rho^\gamma}\, dv / \int e^{-a\rho^\gamma}\, dv$.

It would be interesting to know how C depends on a and x_0, at least when the Ricci curvature is bounded below. For instance, it should be true that, if the Ricci curvature is nonnegative, (16) holds for $\psi = e^{-a\rho^2}$, with a constant equal to $C'a^2$, where C' is independent of x_0 and a. The above also raises the following question: Does (16) hold with $\psi = h_t(x_0, \cdot)$ (i.e., the heat kernel) for some fixed $0 < t < +\infty$ and $x_0 \in M$?

10. Sobolev inequalities

In this section, we prove the Sobolev inequalities on balls which are the main ingredient for most of the results obtained in this paper. We also discuss some related inequalities. The proofs rest mainly on the basic kernel estimates described in §2. We also use abstract semigroup techniques developed by N. Varopoulos in [40]. Namely, we use the following proposition, which can easily be deduced from [40, Theorem 1].

Proposition 10.1. *Let $T_t = e^{-tA}$ be a symmetric submarkovian semigroup, and assume that*

$$\|T_t\|_{1\to+\infty} \leq C_0 t^{-m/2}, \qquad 0 < t \leq t_0,$$

for some $m > 2$. *Then, there exists* C, *depending only on* m, *such that*

$$\|f\|_{2m/(m-2)} \le C_0^{1/m}(C\|A^{1/2}f\|_2 + t_0^{-1/2}\|f\|_2), \qquad f \in \mathscr{D}(A^{1/2}).$$

Note that, in fact, Varopoulos's result is that the above two properties are equivalent (with different constants C_0) (see [40] and [42]). We will also need a more general result. Let $\{R_t, t \in]0, +\infty[\}$ be a family of operators acting on the spaces L^p, $1 \le p \le +\infty$, and consider the operator R defined formally by

$$R = \int_0^{+\infty} R_t \frac{dt}{t}.$$

Proposition 10.2. *Suppose that* $\|R_t\|_{1\to+\infty} \le C_0 t^{a/2-m/2}$, $0 < t < +\infty$, *for some fixed* $0 < a < m$.

(i) *If, for all* $1 < q < m/a$, *there exists* C_q *such that* $\|R_t\|_{q\to q} \le C_q t^{a/2}$, $0 < t < +\infty$, *then, for all* $1 < p < m/a$, *there exists* C, *depending on* a, m, p, *and the* C_q *'s such that*

$$\|Rf\|_{mp/(m-ap)} \le C C_0^{a/m}\|f\|_p, \qquad f \in L^p.$$

(ii) *If* $\|R_t\|_{1\to 1} \le C_1 t^{a/2}$, $0 < t < +\infty$, *then there exists* C *depending on* a, m, *and* C_1, *such that*

$$v(\{|Rf| > s\}) \le (C C_0^{a/m} s^{-1}\|f\|_1)^{m/(m-a)}, \qquad f \in L^1.$$

We sketch the proof which is adapted from the arguments in [40]. We first prove (ii). Let $f \in L^1$ be such that $\|f\|_1 = 1$, and write $Rf = f^T + f_T$ with $f^T = \int_0^T R_t f t^{-1}\, dt$, $T > 0$. Then

$$|f_T| \le \int_T^{+\infty} \|R_t f\|_\infty \frac{dt}{t} \le C C_0 T^{-(m-a)/2}.$$

Given any fixed $s > 0$, we choose T such that $s/2 = C C_0 T^{-(m-a)/2}$. For that choice of T, we have

$$v(\{|Rf| > s\}) \le v(\{|f^T| > s/2\}) \le 2s^{-1}\|f^T\|_1.$$

Since

$$\|f^T\|_1 \le \int_0^{+\infty} \|R_t f\|_1 \frac{dt}{t} \le C T^{a/2},$$

$$v(\{|Rf| > s\}) \le C s^{-1} T^{a/2} = C' C_0^{a/(m-a)} s^{-m/(m-a)},$$

which proves (ii). Assertion (i) is obtained by first proving the L^p version of (ii), and then applying Marcienkiewicz's interpolation theorem (see

[42]). Note that Proposition 10.1 can be deduced from Proposition 10.2 by writing

$$f = T_{t_0} f - \int_0^{t_0} \partial_t T_t f \, dt = T_{t_0} f + \int_0^{t_0} A^{1/2} T_t A^{1/2} f \, dt,$$

and setting $R_t = t\mathscr{X}_{t_0}(t) A^{1/2} T_t$, where $\mathscr{X}_{t_0}(t) = 1$, if $0 \le t \le t_0$, and 0 otherwise.

Proof of Theorem 3.1. Fixing a ball $B \subset M$, we need to prove the inequality

$$(17) \quad \|f\|_{2n/(n-2)} \le e^{C(1+\sqrt{K}r)} V^{-1/n} r (\|\nabla f\|_2 + r^{-1} \|f\|_2), \qquad f \in \mathscr{C}_0^\infty(B),$$

if $n > 2$ (and with n replaced by $n' > 2$ if $n \le 2$). Let $H_{B,t}^D$ be the heat flow semigroup associated with Δ and the Dirichlet boundary condition on B, and let $h_{B,t}^D$ be its kernel. Then

$$h_{B,t}^D(z, y) \le h_t(z, y) \le h_t^{1/2}(z, z) h_t^{1/2}(y, y),$$

where h_t is the heat kernel on M. By Lemma 2.1, and the volume estimate (2), we obtain

$$h_{B,t}^D(z, y) \le e^{C(1+\sqrt{K}r)} V^{-1} r^n t^{-n/2}, \qquad 0 < t \le r^2,$$

which can be rephrased as

$$\|H_{B,t}^D\|_{1\to\infty} \le e^{C(1+\sqrt{K}r)} V^{-1} r^n t^{-n/2}, \qquad 0 < t \le r^2.$$

Thus, (17) follows after applying Proposition 10.1 to $H_{B,t}^D$ (if $n \le 2$, note that $(r/\sqrt{t})^n \le (r/\sqrt{t})^n$ for any $m > 2$ and $0 < t \le r^2$). We now pass to the proof of L^p-versions of Theorem 3.1.

Theorem 10.3. *If $1 \le p < n$, then there exists C, depending on n and p, such that, for any ball $B \in M$, we have*

$$\|f\|_{pn/(n-p)} \le e^{C(1+\sqrt{K}r)} V^{-1/n} r (\|\nabla f\|_p + r^{-1} \|f\|_p), \qquad f \in \mathscr{C}_0^\infty(B).$$

If $p \ge n$, then the above inequality holds with n replaced by any $n' > p$.

Proof. By classical arguments, it is enough to prove the above inequality for $p = 1$. Moreover, by the famous co-area formula

$$\int |\nabla f| \, dv = \int_0^{+\infty} v_{n-1}(\{|f| > s\}) \, ds,$$

the inequality

$$\|f\|_{n/(n-1)} \le C_0 (\|\nabla f\|_1 + r^{-1} \|f\|_1), \qquad f \in \mathscr{C}_0^\infty(B),$$

follows from the apparently weaker estimate

$$(18)\quad v(\{|f|>s\}) \le (C_0 s^{-1}(\|\nabla f\|_1 + r^{-1}\|f\|_1))^{n/(n-1)}, \qquad f\in\mathscr{C}_0^\infty(B).$$

Thus, we are left with the task of proving (18) with $C_0 = e^{C(1+\sqrt{K}r)}V^{-1/n}r$. To simplify notation, denote by $Q_t = P^D_{B,t}$ the Poisson semigroup associated with Δ and the Dirichlet boundary condition on $\frac{3}{2}B$ (Q_t is obtained from $H^D_{B,t}$ by subordination). By the same line of reasoning that we applied above to the heat semigroup, we obtain

$$(19)\qquad q_t(z,y) \le e^{C(1+\sqrt{K}r)}V^{-1}r^n t^{-n}, \qquad 0<t\le r,$$

where q_t is the kernel of Q_t, or equivalently

$$(20)\qquad \|Q_t\|_{1\to\infty} \le e^{C(1+\sqrt{K}r)}V^{-1}r^n t^{-n}, \qquad 0<t\le r.$$

For $f\in\mathscr{C}_0^\infty(B)$, we have $f=\int_0^r s\partial_s^2 Q_s f\,ds - r\partial_r Q_r f + Q_r f$. By interpolation, from (20) we deduce that

$$\|Q_t f\|_{1\to n/(n-1)} \le e^{C(1+\sqrt{K}r)}V^{-1/n}rt^{-1}, \qquad 0<t\le r,$$

and, by holomorphy of the submarkovian semigroup Q_t on $L^{n/(n-1)}$ (see [37]), we also have

$$t\|\partial_t Q_t\|_{1\to n/(n-1)} \le e^{C(1+\sqrt{K}r)}V^{-1/n}rt^{-1}, \qquad 0<t\le r.$$

Let χ be the characteristic function of the ball B. The above inequality reduces the proof of (18) to the study of the operator J defined by

$$Jf = \chi\int_0^r t\partial_t^2 Q_t f\,dt.$$

More precisely, we need to prove

$$v(\{|Jf|>s\}) \le (e^{C(1+\sqrt{K}r)}V^{-1/n}r\|\nabla f\|_{1/s})^{n/(n-1)}.$$

Remark that

$$|Jf| = \chi\int_0^r tQ_t\Delta f\,dt \le \int_0^{+\infty} R_t|\nabla f|\frac{dt}{t} = R|\nabla f|,$$

where we have set

$$R_t f(z) = t^2\chi(z)\int |\nabla_y q_t(z,y)|\chi(y)f(y)\,dv(y), \qquad 0<t\le r,$$

and $R_t=0$ otherwise. Using Cheng-Yau's estimate (4), we obtain

$$\chi(z)|\nabla_y q_t(z,y)|\chi(y) \le C(t^{-1}+\sqrt{K})q_t(z,y),$$

and therefore

$$\|R_t\|_{1\to 1} \le C(1+\sqrt{K}r)t, \qquad 0 < t < +\infty,$$

which together with (19) implies

$$\|R_t\|_{1\to\infty} \le e^{C(1+\sqrt{K}r)} V^{-1} r^n t^{1-n}, \qquad 0 < t < +\infty.$$

The last two estimates allow us to use Proposition 10.2 to get

$$v(\{R|\nabla f| > s\}) \le (e^{C(1+\sqrt{K}r)} V^{-1/n} r \|\nabla f\|_1 / s)^{n/(n-1)}.$$

This is the desired example, and ends the proof of Theorem 10.3. q.e.d.

With the help of the relevant Poincaré estimates from §9, the above theorem yields

Theorem 10.4. *If $1 \le p < n$, then there exists C, depending only on n and p, such that, for $0 < \delta < 1$ and any ball $B \in M$, we have*

$$\|f - f_B\|_{q,\delta B} \le (1-\delta)^{-1} e^{C(1+\sqrt{K}r)} v^{-1/n} r \|\nabla f\|_{p,B}, \qquad f \in \mathscr{C}^\infty(B),$$

where $q = np/(n-p)$. When $B \ne M$, we also have

$$\|f\|_q \le e^{C(1+\sqrt{K}r)} V^{-1/n} r \|\nabla f\|_p, \qquad f \in \mathscr{C}_0^\infty(\tfrac{1}{2}B).$$

If $p > n$, then the above inequality holds with n replaced by any fixed $n' > p$.

Note that, when $p = 1$, the inequalities of Theorems 10.3 and 10.4 are equivalent to some isoperimetric inequalities. These results can be complemented when $p \ge n$. We introduce the notation

$$\Lambda_{\gamma,B}(f) = \sup\{|f(x) - f(y)|/\rho(x,y)^\gamma,\ x, y \in B\}, \qquad \Lambda_\gamma = \Lambda_{\gamma,M}.$$

Theorem 10.5. *Let $p > n$ and set $\gamma = 1 - n/p$. Then there exists C, depending on n and p, such that, for all $B \subset M$, we have*

$$\Lambda_\gamma(f) \le e^{C(1+\sqrt{K}r)} V^{-1/p} r^{n/p} (\|\nabla f\|_p + r^{-1}\|f\|_p), \qquad f \in \mathscr{C}_0^\infty(B).$$

For $0 < \delta < 1$, we also have

$$\Lambda_{\gamma,\delta B}(f) \le (1-\delta)^{-1} e^{C(1+\sqrt{K}r)} V^{-1/p} r^{n/p} \|\nabla f\|_{p,B}, \qquad f \in \mathscr{C}^\infty(B),$$

and if $B \ne M$, then

$$\Lambda_\gamma(f) \le e^{C(1+\sqrt{K}r)} V^{-1/p} r^{n/p} \|\nabla f\|_p, \qquad f \in \mathscr{C}_0^\infty(\tfrac{1}{2}B).$$

Proof. Fix $B = B(x, r) \subset M$ and $f \in \mathscr{C}_0^\infty(B)$. We are going to estimate $\Lambda_\gamma(f)$ by $\|\nabla f\|_p$ and $\|f\|_p$ for a fixed $p > n$ and $\gamma = 1 - n/p$.

Using the notation of the proof of Theorem 10.3, we write for $y, z \in B$ and $0 < t \le r$,

$$|f(z) - f(y)| \le 2 \int_0^t \|\partial_s Q_s f\|_{\infty, B}\, ds + |Q_t f(z) - Q_t f(y)|.$$

The first term of the right-hand side of the above inequality can be estimated as follows:

$$\|\partial_s Q_s f\|_{\infty, B} \le \int_s^r \|\partial_s^2 Q_s f\|_{\infty, B}\, ds + \|\partial_r Q_r f\|_{\infty, B}, \qquad 0 < s \le r.$$

By the arguments of the preceding proof, we obtain

$$\|\partial_s^2 Q_s f\|_{\infty, B} \le s^{-2} \|R_s |\nabla f|\, \|_{\infty, B} \le C(K, r)^{1/p} s^{-1-n/p} \|\nabla f\|_p,$$

where $C(K, r) = e^{C(1+\sqrt{K}r)} V^{-1} r^n$. We also have

$$\|\partial_r Q_r f\|_{\infty, B} \le C(K, r)^{1/p} r^{-1-n/p} \|f\|_p,$$

and therefore

$$\int_0^t \|\partial_s Q_s f\|_{\infty, B}\, ds \le C(K, r)^{1/p} (t^{1-n/p} \|\nabla f\|_p + t r^{-1-n/p} \|f\|_p).$$

We now look at $|Q_t f(z) - Q_t f(y)|$, and set $B' = \frac{5}{4} B \subset \frac{3}{2} B$ and $\rho = \rho(z, y)$. Since $z, y \in B$, we have

$$|Q_t f(z) - Q_t f(y)| \le C \rho \|\nabla Q_t f\|_{\infty, B'}$$

and

$$\nabla Q_t f = \int_t^r (s - t) \partial_s^2 \nabla Q_s f\, ds + (t - r) \partial_r \nabla Q_r f + \nabla Q_r f.$$

Repeatedly using Cheng-Yau's estimate (4) yields

$$|\nabla_1 \nabla_2 q_s(x_1, x_2)| \le C(s^{-1} + \sqrt{K})^2 q_s(x_1, x_2)$$

and

$$|\partial_s \nabla_1 q_s(x_1, x_2)| \le C(s^{-1} + \sqrt{K})^2 q_s(x_1, x_2)$$

for $0 < s \le r$ and $x_1, x_2 \in B'$. To see this, we apply (4) to show that there exists C for which

$$w(s, \xi) = \partial_s q_s(\xi, \cdot) + C(\tau^{-1} + \sqrt{K}) q_s(\xi, \cdot)$$

is a positive solution of $(-\partial_s^2 + \Delta) w = 0$ in $]\tau/2, 3\tau/2[\times \frac{11}{8} B$, $0 < \tau \le r$, and apply (4) again. This proves the second of the above estimates, and the first can be obtained by a similar argument. Following the arguments of the proof of Theorem 10.3 again, we write

$$\partial_s^2 \nabla Q_s f(x_1) = \int \nabla_{1g} (\nabla_2 q_s(x_1, x_2), \nabla f(x_2))\, dv(x_2),$$

so that

$$\|\nabla Q_t f\|_{\infty, B'} \le C(K, r)^{1/p}(t^{-n/p}\|\nabla f\|_p + r^{-1-n/p}\|f\|_p).$$

Summing up the above estimates, we get

$$|f(z) - f(y)| \le C(K, r)^{1/p}((t^{1-n/p} + \rho t^{-n/p})\|\nabla f\|_p + (t+\rho)r^{-1-n/p}\|f\|_p),$$

which yields $|f(z) - f(y)| \le \rho^{1-n/p} C(K, r)^{1/p}(\|\nabla f\|_p + r^{-1}\|f\|_p)$ for $t = \rho \le r$. With the help of the relevant Poincaré inequalities, we end the proof of Theorem 10.5.

Finally, in the limit case $p = n$, arguments similar to the above ones allow us to deduce the following result from an abstract semigroup theorem of Lohoué (see [29], Theorem 1]).

Theorem 10.6. *There exist two constants C and C', depending only on n, such that, for all $B \subset M$ and all open $\Omega \subset B$, we have*

$$\int \{e^{(|f|/\beta(\|\nabla f\|_n + r^{-1}\|f\|_n))^{n/(n-1)}} - 1\}\, dv \le C' v(\Omega), \qquad f \in \mathscr{C}_0^\infty(\Omega),$$

where $\beta = e^{C(1+\sqrt{K}r)} V^{-1/n} r$. Moreover, for $0 < \delta < 1$, we have

$$\int_{\delta B} \{e^{((1-\delta)|f - f_B|/\beta\|\nabla f\|_{n,B})^{n/(n-1)}}\}\, dv \le C' v(\Omega), \qquad f \in \mathscr{C}^\infty(\Omega).$$

If $B \ne M$, then, for all open $\Omega \subset \frac{1}{2}B$,

$$\int \{e^{(|f|/\beta\|\nabla f\|_n)^{n/(n-1)}} - 1\}\, dv \le C' v(\Omega), \qquad f \in \mathscr{C}_0^\infty(\Omega).$$

When M is compact, or when M is noncompact, but f is compactly supported in a ball B, Theorems 10.4, 10.5, and 10.6 give rather sharp results. In constrast, the statements obtained for functions which are not supported in B are not as sharp; they involve the parameter δ and the smaller ball δB. As indicated in §3, the correct result should be the inequality (7) and, more generally, for $1 \le p < n$,

$$(21) \quad \|f - f_B\|_{np/(n-p), B} \le e^{C(1+\sqrt{K}r)} V^{-1/n} r \|\nabla f\|_{p,B}, \qquad f \in \mathscr{C}^\infty(B),$$

as well as similar statements corresponding to Theorems 10.5 and 10.6. It is worth noticing that, by Proposition 10.1 and the remark following it, (7) is equivalent to the kernel estimate

$$\|h^N_{t,B}\|_{B,\infty} \le e^{C'(1+\sqrt{K}r)} V^{-1} r^n t^{-n}, \qquad 0 < t < +\infty,$$

where $h^N_{B,t}$ is the kernel associated with Neumann boundary condition on B. In general, the available estimates on $h^N_{B,t}$ are interior estimates and hold only inside δB for fixed $\delta < 1$. Hence, these estimates are not strong enough to prove (7) or (21). However, under the additional hypothesis that B has convex boundary, the gradient estimate of Li and Yau extends up to the boundary (see [28, Theorem 1.4]). It follows that, under this convexity hypothesis, the inequalities (7) and (21) hold, and the corresponding statements in Theorems 10.5 and 10.6 can be sharpened as well.

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